

Free algebras of discriminator varieties generated by finite algebras are atomic

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Abstract

We prove that all definable pre-orders are atomic, in a finitely generated free algebra of a discriminator variety of finite similarity type which is generated by its finite members.

A pre-order \leq is a reflexive and transitive relation, $a < b$ abbreviates that $a \leq b$ and $b \not\leq a$. A pre-order \leq is called *atomic* if in each interval the smallest element has a cover, formally if $\forall a < b \exists c [a < c \leq b \wedge \neg \exists x (a < x < c)]$. The name is justified by the fact that a Boolean algebra is atomic iff its natural order is atomic in the above sense. A binary relation \leq in an algebra \mathfrak{F} is called *definable* if it is equationally definable, i.e., if there are terms τ, σ in the language of \mathfrak{F} such that $\mathfrak{F} \models \forall xy [x \leq y \leftrightarrow \tau = \sigma]$. A variety \mathbf{V} is a *discriminator variety* if there is a term σ in its language that in each subdirectly irreducible member of \mathbf{V} is the so-called *switching term*, i.e., $\sigma(x, y, u, v) = u$ if $x = y$ and $\sigma(x, y, u, v) = v$ if $x \neq y$. For this definition of a discriminator variety, and for its basic properties that we will use in the proof, we refer the reader to [2]. Theorem 1 below is a corollary of [1, Theorem 4.1(i)] which states that in an ordered discriminator variety of finite similarity type, each finitely generated residually finite algebra is atomic. (The quoted theorem concerns only orders and not pre-orders, but its proof equally applies to pre-orders.) Here, we give a different, direct proof that might be easier to generalize in certain directions.

Theorem 1 *Let \mathbf{V} be a discriminator variety of finite similarity type that is generated by its finite members as a variety. Any definable pre-order in a finitely generated free algebra of \mathbf{V} is atomic.*

We note that none of the conditions of the theorem can be omitted without affecting its truth, see the discussion at the end of the paper. Theorem 1 has applications in logic, see [8]. Both its statement and proof are a generalization of [4, Thm.2.5.7]. The key ingredients of our proof are that all pre-orders on a finite set are atomic and the discriminator term gives us expressive power.

Proof of Theorem 1. Let \mathfrak{F} be a \mathbf{V} -free algebra freely generated by a finite set X . Assume that the definable binary relation \leq in \mathfrak{F} is a pre-order. Let $\alpha < \beta$ in \mathfrak{F} , i.e., $\alpha \leq \beta$ and $\beta \not\leq \alpha$. We set to finding a cover $\gamma \leq \beta$ of α .

Since $\alpha < \beta$ in \mathfrak{F} and \mathbf{V} is generated by finite algebras, there are a finite subdirectly irreducible $\mathfrak{A} \in \mathbf{V}$ and a homomorphism $h : \mathfrak{F} \rightarrow \mathfrak{A}$ such that $h(\alpha) < h(\beta)$ (in \mathfrak{A}). We may assume that \mathfrak{A} is generated by $h(X) = \{h(x) : x \in X\}$ since a subalgebra of a subdirectly irreducible algebra in a discriminator variety is also subdirectly irreducible. Since X generates \mathfrak{F} , each element of F is of the form $\tau^{\mathfrak{F}}(\bar{x})$ for at least one term τ (where \bar{x} is a sequence of elements from X). Since \mathfrak{A} is also generated by $h(X)$, the homomorphism h is surjective. For each $a \in A$ let $\rho(a)$ denote a term such that $h(\rho(a)) = a$. (ρ stands for “representative”.) Consider the formulas of the following three forms

$$x = \rho(h(x)), \quad f(\rho(a_1), \dots, \rho(a_m)) = \rho(f(a_1, \dots, a_m)), \quad \rho(a) \neq \rho(a')$$

where $x \in X$, f is an m -place operation in \mathfrak{A} , $a_1, \dots, a_m \in A$, and $a, a' \in A$ are distinct. These conditions specify a finite set of formulas because X , the similarity type of \mathfrak{A} , and A are all finite. Thus their conjunction η is a universal formula with $x \in X$ considered to be free variables. Let $\mathfrak{A} \models \eta[h]$ denote that η is true in \mathfrak{A} under the evaluation h of variables, and the same for \mathfrak{B}, k . The formula η describes \mathfrak{A} up to isomorphism, but we will use only $\mathfrak{A} \models \eta[h]$ and the following property of η ,

- (1) Let $k : \mathfrak{F} \rightarrow \mathfrak{B}$ be such that $\mathfrak{B} \models \eta[k]$. Then
 $i = \{\langle h(\tau), k(\tau) \rangle : \tau \in F\}$ is an embedding of \mathfrak{A} into \mathfrak{B} .

Indeed, recall that X generates \mathfrak{F} and so each element of F can be obtained from X by application of a term τ . First one shows by induction on τ that $k(\tau) = k(\rho(h(\tau)))$ for all τ . Indeed, we have $k(x) = k(\rho(h(x)))$ by η , and we have $k(f(\tau_1, \dots, \tau_m)) = f(k(\tau_1), \dots, k(\tau_m)) = f(k(\rho(h\tau_1)), \dots, k(\rho(h\tau_m))) = k(f(\rho(h\tau_1), \dots, \rho(h\tau_m))) = k(\rho(f(h\tau_1, \dots, h\tau_m))) = k(\rho(hf(\tau_1, \dots, \tau_m)))$

by k being a homomorphism, the induction hypothesis, k being a homomorphism, η , and h being a homomorphism, respectively. Thus $i(a) = k\rho(a)$ for all $a \in A$. Hence $i : \mathfrak{A} \rightarrow \mathfrak{B}$ is a homomorphism by k being a homomorphism and the second kind of terms in the definition of η . It is one-to-one by the last term in the definition of η because $a = h(\rho(a))$ for all $a \in A$.

Since we are in a discriminator variety, each universal formula is equivalent to an equation in the subdirectly irreducible algebras. Let δ, ε be terms such that the formula

$$\forall \bar{x} [\eta(\bar{x}) \leftrightarrow \delta = \varepsilon]$$

is valid in the subdirectly irreducible members of \mathbf{V} . Recall that $h(\alpha) < h(\beta)$ in \mathfrak{A} . Since \mathfrak{A} is finite, there is a cover $c \leq h(\beta)$ of $h(\alpha)$ in \mathfrak{A} . Now, we are ready to write up the desired cover term γ of α . Let γ denote

$$\sigma(\delta, \varepsilon, \rho(c), \alpha)$$

where σ is the switching term of the discriminator variety \mathbf{V} . We are going to show that $\gamma \leq \beta$ is a cover of α in \mathfrak{F} .

We begin with showing that $\alpha \leq \gamma \leq \beta$ in \mathfrak{F} . In order to show this, we have to show $\alpha \leq \gamma \leq \beta$ in all subdirectly irreducible elements \mathfrak{B} of \mathbf{V} . Notice that $\alpha \leq \rho(c)$ does not necessarily hold in \mathfrak{F} . Let thus $k : \mathfrak{F} \rightarrow \mathfrak{B} \in \mathbf{V}$ be arbitrary, with \mathfrak{B} subdirectly irreducible. Assume first that $k(\delta) = k(\varepsilon)$. Since k is a homomorphism, $k(\gamma) = k(\sigma(\delta, \varepsilon, \rho(c), \alpha)) = \sigma(k(\delta), k(\varepsilon), k(\rho(c)), k(\alpha))$. Since σ is a switching term in \mathfrak{B} and $k(\delta) = k(\varepsilon)$, we get $k(\gamma) = k(\rho(c))$ and $\mathfrak{B} \models \eta[k]$. By (1) and $h(\alpha) \leq h(\rho(c)) \leq h(\beta)$ then $k(\alpha) \leq k(\rho(c)) \leq k(\beta)$ and we are done. Assume that $k(\delta) \neq k(\varepsilon)$. Then $k(\gamma) = k(\alpha)$ by the definition of γ , so again we are done since $\alpha \leq \beta$ in \mathfrak{F} .

Next we show $\gamma \not\leq \alpha$ in \mathfrak{F} . By the construction of η we have that $\mathfrak{A} \models \eta[h]$. Since \mathfrak{A} is subdirectly irreducible then $\mathfrak{A} \models (\delta = \varepsilon)[h]$, i.e., $h(\delta) = h(\varepsilon)$ by the choice of δ, ε . Thus $h(\gamma) = h(\rho(c))$ by the definition of γ , but $h(\rho(c)) = c$ by the choice of $\rho(c)$. This shows that $h(\gamma) \not\leq h(\alpha)$ since $h(\alpha) < c$ in \mathfrak{A} , and this implies $\gamma \not\leq \alpha$ in \mathfrak{F} .

Finally, we show that γ is a cover of α in \mathfrak{F} . We will show that for all $\tau \in F$ such that $\alpha \leq \tau \leq \gamma$ in \mathfrak{F} we have either $\tau \leq \alpha$ or $\gamma \leq \tau$ in \mathfrak{F} . So, assume $\alpha \leq \tau \leq \gamma$. Then $h(\alpha) \leq h(\tau) \leq h(\gamma)$. Since $h(\gamma) = c$ is a cover of $h(\alpha)$ in \mathfrak{A} , either $h(\tau) \leq h(\alpha)$, or $h(\gamma) \leq h(\tau)$. Let $k : \mathfrak{F} \rightarrow \mathfrak{B}$ be arbitrary with $\mathfrak{B} \in \mathbf{V}$ subdirectly irreducible. Assume $h(\tau) \leq h(\alpha)$. If $k(\delta) \neq k(\varepsilon)$ then $k(\gamma) = k(\alpha)$, so $k(\tau) \leq k(\alpha)$ by $\tau \leq \gamma$. If $k(\delta) = k(\varepsilon)$, then $k(\tau) \leq k(\alpha)$ by $\mathfrak{B} \models \eta[k]$, (1) and $h(\tau) \leq h(\alpha)$. So, in either case we have $k(\tau) \leq k(\alpha)$,

which implies that $\tau \leq \alpha$ in \mathfrak{F} . Assume $h(\gamma) \leq h(\tau)$. If $k(\delta) \neq k(\varepsilon)$ then $k(\gamma) = k(\alpha)$, so $k(\gamma) \leq k(\tau)$ by $\alpha \leq \tau$. If $k(\delta) = k(\varepsilon)$ then $\mathfrak{B} \models \eta[k]$, so by (1) we have $k(\gamma) \leq k(\tau)$ by $h(\gamma) \leq h(\tau)$. Since in either case $k(\gamma) \leq k(\tau)$, we have $\gamma \leq \tau$ in \mathfrak{F} , and we are done. QED

We have the following corollary.

Theorem 2 *Let \mathbf{V} be a discriminator variety that is generated by its finite algebras as a variety. Assume that \mathbf{V} has a Boolean algebra reduct and its similarity type is finite. Then the finitely generated \mathbf{V} -free algebras are atomic.*

Most varieties arising from logic have Boolean algebra reducts, and atomicity of their free algebras corresponds to weak Gödel's incompleteness property holding for the logic, see [9] or [3, 8].

None of the conditions of Theorem 1 can be omitted without affecting its truth. The condition of \mathbf{V} being of finite similarity type is necessary because the free algebra of the class of Boolean algebras with infinitely many constants (of which we do not state any equations) is atomless. The condition of the free algebra generated by finitely many elements is necessary because the infinitely generated free Boolean algebra is atomless. The condition that \mathbf{V} is discriminator is necessary because the variety of 2-dimensional cylindric-relativized set algebras is generated by its finite members but its finitely generated free algebras are not atomic, see [8]. The case is similar for other varieties of relativized algebras in algebraic logic, see [6, 7]. The condition that \mathbf{V} be generated by its finite members is necessary, because varieties of un-relativized algebras in algebraic logic usually do not have atomic free algebras but they are discriminator in the finite-dimensional case. This is the case for the varieties of abstract and representable relation algebras, the abstract and representable finite-dimensional cylindric algebras, diagonal-free 3-dimensional cylindric algebras. See, e.g., [3, 5, 9, 10] or [4, 4.3.32].

References

- [1] Andr  ka, H., J  nsson, B., N  meti, I., Free algebras in discriminator varieties. *Algebra Universalis* 28 (1991), 401-447.
- [2] Burris, S., Sankappanavar, H. P., A course in Universal Algebra. Springer-Verlag, 1981.

- [3] Gyenis, Z., On atomicity of free algebras of certain cylindric-like varieties. *Logic Journal of IGPL* 19,1 (2011), 44-52.
- [4] Henkin, L., Monk, J. D., Tarski, A., *Cylindric algebras, Parts I-II*. North-Holland, Amsterdam, 1971 and 1985.
- [5] Hirsch, R., Hodkinson, I., *Relation algebras by games*. North Holland, Amsterdam, 2002.
- [6] Khaled, M., Weak Gödel's incompleteness property for some decidable versions of the calculus of relations. *arXiv:1511.01383*, 2015.
- [7] Khaled, M., Weak Gödel's incompleteness property for some decidable versions of first order logic. *arXiv:1511.05221*, 2015.
- [8] Khaled, M., Gödel's incompleteness properties and the guarded fragment: an algebraic approach. PhD Dissertation, Central European University, Department of Mathematics and its Applications, Budapest Hungary. 2016.
- [9] Németi, I., Logic with three variables has Gödel's incompleteness property - thus free cylindric algebras are not atomic. Preprint No 49/85, Math. Inst., Budapest, 1985. <http://www.renyi.hu/~nemeti/NDis/NPrep85.pdf>
- [10] Tarski, A., Givant, S. R., *A formalization of set theory without variables*. Colloquium Publications Vol 41, American Mathematical Society, Providence, R. I., 1987.

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